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FILTRATION TOWARD DRAINS WITH SHIELDED SLOPES

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The plane problem of steady nonsymmetric two-sided inflow of ground water to a drain of trapezoidal cross section with shielded slopes in a zero-head stratum with an impervious base is solved in hydrodynamic formulation by conformal mapping. A system of equations is obtained for the unknown parameters. The qualitative analysis of the derived solution shows that the branches of the depression curve tend to become parabolic with increasing distance from the channel, and the flow is defined with ever greater exactness by relationships of the hydraulic theory of zero-head filtration. When the waterproof strata is moved downward to infinity, the parabolic asymptotics of the depression curve branches is replaced by a logarithmic one. In the case of one-sided inflow (of water) (in the presence of a watertight stratum) the opposite branch of the depression curve has a horizontal asymptote which it approaches in conformity with an exponential law.

Slopes of drainage channels of trapezoidal cross section have to be sometimes shielded because of insufficient firmness of the ground. In such cases drainage takes place through the channel floor. Such flow is diagrammatically shown in Fig. 1 for the case in which the layer of soil undergoing drainage is supported by a waterproof stratum, and the drainage rates from left and right are, generally, different, i.e. the flow is nonsymmetric.

We introduce the complex potential $\omega = \varphi + i\psi$, where φ is the potential of filtration rate and ψ is the stream function. If we assume that $\varphi|_{CC'} = \psi|_{A'B'C'} = 0$, then region ω is represented by the half-band with the slit AEA' (Fig. 2, a) to whose tip corresponds in region z a neutral point lying on line ED of separation of streams; the rate of flow at that point is zero. We shall deal below with reduced parameters z and ω related to the similar physical quantities z' and ω' by formulas

$$z = z' / l, \qquad \omega = \omega' / kl \tag{1}$$

where l is the half-width of the channel floor and k is the filtration coefficient.

We denote by Q the reduced rate of filtration through the channel floor per unit of its length per unit of time. The rates of flow from the left and right are denoted, respectively, by $(1-\varkappa)Q$ and $\varkappa Q$ $(0<\varkappa<1)$. Mapping region ω onto the half-plane $\operatorname{Im}\zeta \geqslant 0$ (Fig. 2, b), we obtain

$$\omega = -M\int_1^{\zeta}\Omega(\tau)\,d\tau + iQ, \quad \Omega(\tau) = \frac{1}{(a-\tau)(a'+\tau)\sqrt{\tau^2-1}} = \frac{\Omega_a(\tau)}{a-\tau} \tag{2}$$

The term $\sqrt{\tau^2-1}$ implies the branch that is positive when $\tau > 1$.

Equating the increments of function ω and of the integral in formula (2), in passing through the points A and A', we obtain

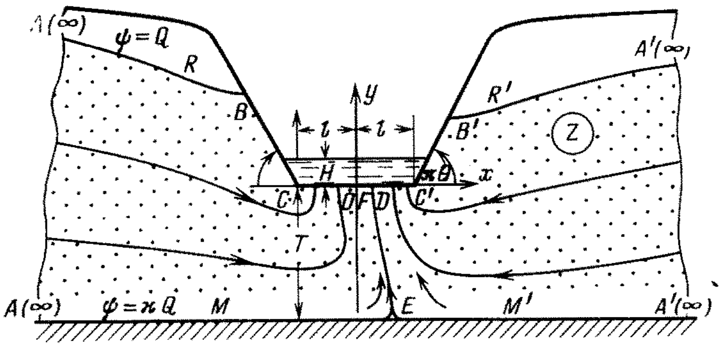

Fig. 1

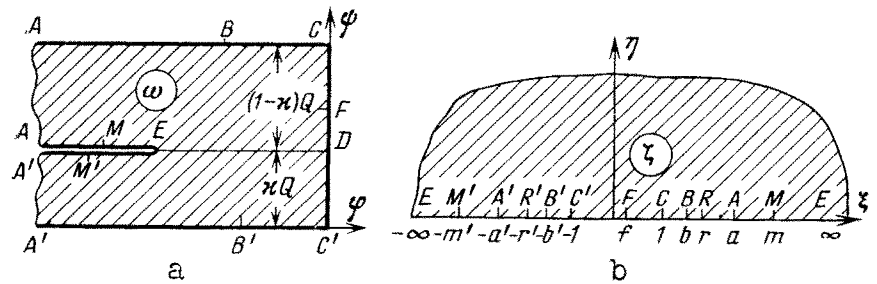

Fig. 2

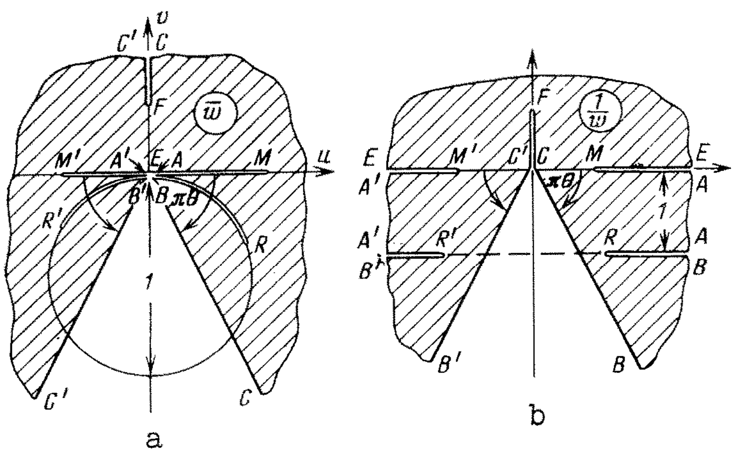

Fig. 3

$$M = (1-\varkappa)\frac{Q}{\pi}[\Omega_a(a)]^{-1} = (1-\varkappa)\frac{Q}{\pi}(a+a')\sqrt{a^2-1} = \varkappa\frac{Q}{\pi}(a+a')\sqrt{a'^2-1} \tag{3}$$

from which

$$a'^2 = a^2 + (a^2-1)(\lambda^2-1), \quad \lambda = (1-\varkappa)/\varkappa \tag{4}$$

The calculation of the integral in (2) yields for ω along boundary sections the following formulas:

$$-\frac{\pi}{Q}\omega(\zeta) = \begin{cases} -i[(1-\varkappa)\arccos\alpha + \varkappa\arccos(-\beta)], & -1 \leqslant \zeta \leqslant 1 \\ (1-\varkappa)\operatorname{arch}\alpha + \varkappa\operatorname{arch}(-\beta), & -a' < \zeta < -1 \\ (1-\varkappa)\operatorname{arch}(-\alpha) + \varkappa\operatorname{arch}\beta - i\pi, & 1 < \zeta < a \\ (1-\varkappa)\operatorname{arch}\alpha + \varkappa\operatorname{arch}\beta - i\varkappa\pi, & \zeta < -a',\ \zeta > a \end{cases} \tag{5}$$

$$\alpha = \frac{1-a\zeta}{a-\zeta}, \quad \beta = \frac{1+a'\zeta}{a'+\zeta}$$

Let us consider the complex filtration rate $w = w_x - iw_y = d\omega / dz$ and its conjugate physical filtration rate $\overline{w} = w_x + iw_y$. First, we note that $|w|_{C,C'} = \infty$, $|w|_{A,A',E} = 0$. The quantity $|w|$ attains its minimum at some point F of the channel floor CC'. Assuming that within sections CF and CF' its variation is monotonic we find that the semi-infinite slit CFC' along the positive imaginary semiaxis (Fig. 3, a) corresponds in plane w to segment CC'. The similar assumption about the behavior of $|w|$ along sections AE and $A'E$ shows that the watertight bearing stratum is represented in the plane $\overline{w}$ by the slit $AMEM'A'$ along the real axis.

The assumption that quantity $|w|$ varies monotonically along the slopes implies that $w_{B,B'} = 0$, with the tangents of both branches of the depression curve at the points B and B' emerging horizontally through the slopes.

In fact, if, for instance, the second of possible variants of behavior of flow rate $w_B = \sin\pi\theta e^{-i\pi\theta}$ was realized at point B with the left-hand branch of the depression curve tangent to the slope, then the inequality $\partial p / \partial s = \sin\pi\theta - |w| < 0$ for $1 \leqslant \zeta < b$ must be satisfied when the indicated assumption and formulas

$$|w|_{BC} = -\partial\varphi / \partial s = -\partial p / \partial s - \partial y / \partial s = -\partial p / \partial s + \sin\pi\theta$$

where p is the pressure reduced with respect to the specific weight of water and s is the basis vector directed downward along the slope, are satisfied.

Thus the realization of the considered possibility requires the artificial creation and maintenance throughout the flooded section of the slope of vacuum that would increase with depth, reaching its maximum at point C. However, drains work under conditions of free inflow with the pressure at section CC' equal to the sum of atmospheric pressure and the water head in the channel.

The region of function $1/w$ which results in the inversion of the hodograph region with respect to circle $|w| = 1$, is a polygon with rectilinear boundaries (Fig. 3, b). Its conformal mapping onto the half-plane $\operatorname{Im}\zeta \geqslant 0$ (Fig. 2, b) yields

$$\frac{1}{w(\zeta)} = \frac{dz}{d\omega} = \frac{1}{w_M} + N\int_m^\zeta W(\tau)\,d\tau \tag{6}$$

$$W(\tau) = \frac{P(\tau)}{(\tau - a)(\tau + a')(\tau^2 - 1)^{1/2-\theta}[(\tau - b)(\tau + b')]^{1+\theta}} = \frac{W_a(\tau)}{a - \tau}$$

$$P(\tau) = (\tau - m)(\tau + m')(\tau - r)(\tau + r')(\tau - f) = \sum_{n=0}^{5} c_n \tau^n \quad (c_5 = 1)$$

We select in the considered half-plane $\operatorname{Im} \zeta \geqslant 0$ that of the branches of function $W(\zeta)$ which is positive for $\zeta > m$, then $N > 0$, since $1 / w > 1 / w_M$ when $\zeta > m$. The representation of function $W(\tau)$ in terms of function $W_a(\tau)$ which is analytic at point $\tau = a$ (see (2)) will be used below. Note that $W_a(\tau) > 0$ for $1 < \tau < m$.

On the basis of (6) we have

$$z = z(\zeta_0) + \int_{\zeta_0}^{\zeta} \frac{1}{w(\tau)} \frac{d\omega(\tau)}{d\tau} d\tau =$$

$$z(\zeta_0) + \frac{\omega(\zeta) - \omega(\zeta_0)}{w(\zeta_0)} + N \int_{\zeta_0}^{\zeta} W(\tau)[\omega(\zeta) - \omega(\tau)] d\tau$$

We select here $\zeta_0 = 1$ and, since $z(1) = -1$ and $w(1) = \infty$, obtain

$$z(\zeta) = -1 + N \int_1^{\zeta} W(\tau)[\omega(\zeta) - \omega(\tau)] d\tau \tag{7}$$

Formulas (7) and (2) represent the parametric solution of the problem that (for specified Q and $\varkappa$) contains 11 unknown parameters: M, N, a, a', b, b', m, m', r, r' and f. Two of these: M and a' are expressed in terms of parameter a in formulas (3) and (4) derived from the analysis of region ω; the remaining nine relationships are established with the use of known geometric elements of regions $1 / w$ and z.

We pass into the plane $1 / w$ from section ME to section EM' along the semicircle Γ_R of fairly large radius R whose center is at the coordinate origin and along which $\tau = Re^{i\gamma}$. Using (6) and the expansion of function $W(\tau)$ in the neighborhood of point $\tau = \infty$ in the Taylor series [1]

$$W(\tau) = 1 + \frac{c}{\tau} + \frac{1}{\tau^2} O(1), \quad c = c_4 + a - a' + (1 + \theta)(b - b')$$

we obtain

$$\frac{1}{w(-R)} - \frac{1}{w(R)} = N \int_{\Gamma_R} W(\tau) d\tau = N \int_0^{\pi} \left[iRe^{i\gamma} + ic + \frac{1}{R} ie^{-i\gamma} O(1) \right] d\gamma =$$

$$-N(2R - i\pi c) + O\left(\frac{1}{R}\right)$$

Passing here to the limit $R \to \infty$ and taking into consideration that the increment of the imaginary part of function $1 / w$ is zero, we conclude that

$$c = a - a' + (1 + \theta)(b - b') - m + m' - r + r' - f = 0 \tag{8}$$

The imaginary part of function $1 / w$ has at points A and A' increments equal unity. On the strength of this we establish two more relationships

$$\pi N W_a(a) = \pi N \frac{|P(a)|}{(a + a')(a^2 - 1)^{1/2-\theta}[(a - b)(a + b')]^{1+\theta}} = 1 \tag{9}$$

$$\pi N \frac{|P(a')|}{(a + a')(a'^2 - 1)^{1/2-\theta}[(a' - b')(a' + b)]^{1+\theta}} = 1 \tag{10}$$

It follows from Fig. 3, b that the imaginary part of function $1/w$ when traversing section CB and passing over to section AB varies from 0 to -1, i.e.

$$\operatorname{Im} N \int_1^{\zeta} W(\tau)\, d\tau = -1, \quad b < \zeta < a$$

The integral in the left-hand side of the equality does not exist in the conventional meaning owing to the singulatity of function $W(\tau)$ at point $\tau = b$. We determine it by excluding from the integration interval the small neighborhood $(b - \varepsilon,\ b + \varepsilon)$ of point $\tau = b$, as is done in similar cases [1], and substituting for it the semicircle Γ_b of small radius ε. Successive transformations with passing to the limit $\varepsilon \to 0$ yield the equation

$$N \sin \pi\theta \left[\int_1^b \frac{W_b(\tau) - W_b(b)}{(b-\tau)^{1+\theta}}\, d\tau - \frac{W_b(b)}{\theta\,(b-1)^{\theta}} \right] = 1 \tag{11}$$

$$W_b(\tau) = |W(\tau)(\tau - b)^{1+\theta}|$$

We similarly establish one more formula that determines the increment of the imaginary part of function $1/w$ from point C' to section $A'B'$. We have

$$N \sin \pi\theta \left[\int_1^{b'} \frac{W_{b'}(\tau) - W_{b'}(b')}{(b'-\tau)^{1+\theta}}\, d\tau - \frac{W_{b'}(b')}{\theta\,(b'-1)^{\theta}} \right] = 1 \tag{12}$$

$$W_{b'}(\tau) = |W(\tau)(\tau + b')^{1+\theta}|$$

The condition for $CF = C'F$ in the plane $1/w$ of the form

$$\int_{-1}^{1} |W(\tau)| \operatorname{sign}(\tau - f)\, d\tau = 0 \tag{13}$$

Owing to the analyticity of function $1/w$, formula (13) is the corollary of relationships (8) — (12), and can be used as a test of calculations.

Let us revert to region z. Setting in (7) $\zeta = -1$ and $z = 1$, and using (13), we obtain

$$N \int_{-1}^{1} W(\tau)\, \omega(\tau)\, d\tau = 2 \tag{14}$$

The derivation of the following two equations is based on the dependence $\varphi = -(p + y) + p_a + H$ (p_a is the reduced atmospheric pressure and H is the depth of the water layer in the channel). For points B and B' at which $p = p_a - h_k$ [2] (h_k is the height of capillary rise of water in the soil) that dependence of the basis of (7) is of the form

$$N \sin \pi\theta \int_1^b \frac{W_b(\tau)\, \Phi(\tau, b)}{(b-\tau)^{1+\theta}}\, d\tau = -\varphi(b) + H + h_k \tag{15}$$

$$(\Phi(s_1, s_2) = \varphi(s_1) - \varphi(s_2))$$

$$N \sin \pi\theta \int_1^{b'} \frac{W_{b'}(\tau)\, \Phi(-\tau, -b')}{(b'-\tau)^{1+\theta}}\, d\tau = -\varphi(-b') + H + h_k \tag{16}$$

Functions $W_b(\tau)$ and $W_{b'}(\tau)$ are determined in (11) and (12). Function $\varphi(\tau)$ appearing in (14) — (16) is represented along the related integration intervals by formulas (5); owing to its analyticity at points B and B', integrals in (15) and (16), as well as in (11) and (12) are convergent (with singularities of integrands of order $\theta < 1$).

The closing equation for parameters is based on the specification of the height T of the channel floor above the watertight stratum. Since along the latter $y = -T$, hence in conformity with (7)

$$\operatorname{Im} N \int_1^{\zeta} W(\tau)\,[\omega(\tau) - \omega(\zeta)]\,d\tau = -T; \quad a < \zeta < \infty, \quad -\infty < \zeta < -a'$$

We select segment $[1,\ a + \varepsilon]$ of the real axis of plane ζ as the integration path and substitute in it the neighborhood $(a - \varepsilon,\ a + \varepsilon)$ of point $\tau = a$, where the integral is divergent, the semicircle Γ_a of radius ε. In region z this corresponds to the passing of slope CB and of some section of the left-hand branch of the depression curve adjoining the slope and passing to the watertight stratum. The transformations of the integral finally yield the sought equation which in the previously used notation may be presented in the form

$$\pi \operatorname{ctg} \pi\theta + \frac{W_b(b)}{W_a(a)\,\theta\,(a - b)^{\theta}} + \int_b^a \frac{\Lambda(\tau)\,d\tau}{(\tau - b)^{1+\theta}(a - \tau)} = \frac{\pi\,[T + H + h_k - \varphi(b)]}{(1 - \varkappa)\,Q} \tag{17}$$

$$\Lambda(\tau) = \frac{[W_b(b) - W_b(\tau)]\,(a - \tau)}{W_a(a)} - \frac{\Omega_a(\tau)\,(\tau - b)^{1+\theta}}{\Omega_a(a)} =$$

$$\frac{W_b(b)}{W_a(a)}(a - \tau) + \left[\frac{W_a(\tau)}{W_a(a)} - \frac{\Omega_a(\tau)}{\Omega_a(a)}\right](\tau - b)^{1+\theta}$$

Note that the integrand in (17) is analytic when $\tau = a$, while for $\tau = b$ it has an integrable singularity of order θ.

Integration with respect to ζ in the opposite direction, i.e. from 1 to $-a' - \varepsilon$, which in region z corresponds to passing from point C over the right-hand side branch of the depression curve to the watertight stratum, results in a formula which can be derived from (17) by transposing parameters a and a', b and b', and substituting $\varkappa$ for $1 - \varkappa$. A similar transposition transforms formula (9) into (10), (11) into (12), and (15) into (16) or vice versa. Owing to the analyticity of function $z(\zeta)$ this relationship is a corollary of Eq. (17) and conditions $\operatorname{Im} z'_{\zeta} = 0$ on CC' and AA' It can be used for testing calculations.

Formulas (8) — (12) and (14) — (17) form a system of equations relative to parameters. The direct determination of these is very difficult owing to the complexity of the system. In calculations it is possible to specify, for example, parameters a, b and b' and determine parameter a' with the use of equality (4). Then, by eliminating the quantity N from the system of Eqs. (8) — (12) with the use of (14), we uniquely determine coefficients $c_0, \ldots, c_4$ of the polynomial $P(\tau)$. In such semi-inverse procedure the three geometric parameters T, y_B and $y_{B'}$ are "floating"; their values are obtained as a result of calculations.

The complex parameteric equation of the left-hand branch AB of the depression curve, obtained from formula (7), is of the form

$$z(\zeta) = -Ne^{-i\pi\theta} \int_1^b \frac{W_b(\tau)\,\Phi(\tau, b)}{(b - \tau)^{1+\theta}}\,d\tau - \Phi(b, \zeta)\left[\operatorname{ctg}\pi\theta + N\frac{W_b(b)}{\theta\,(\zeta - b)^{\theta}} - i\right] -$$
$$N \int_b^{\zeta} \frac{W_b(b)\,\Phi(b, \zeta) - W_b(\tau)\,\Phi(\tau, \zeta)}{(\tau - b)^{1+\theta}}\,d\tau, \quad b \leqslant \zeta < a \tag{18}$$

The reciprocal transposition of parameters a and a' and b and b' in (18) yields the equation of the right-hand branch of $A'B'$ when $b' \leqslant \zeta < a'$.

Let us investigate the properties of flow in the neighborhood of point A remote from the channel. Taking into account (2) and (6) we can write

$$\frac{d^2z}{d\omega^2} = \frac{d}{d\omega}\left(\frac{1}{w(\zeta)}\right) = \left(\frac{d\omega}{d\zeta}\right)^{-1} \frac{d}{d\zeta}\left(\frac{1}{w(\zeta)}\right) = -\frac{NW(\zeta)}{M\Omega(\zeta)} = -\frac{NW_a(\zeta)}{M\Omega_a(\zeta)}$$

Using (3) and (9) with $\zeta \approx a$, we obtain

$$\frac{d^2z}{d\omega^2} \approx -\frac{NW_a(a)}{M\Omega_a(a)} = -\frac{1}{(1-\varkappa)Q}, \quad z \approx -\frac{\omega^2}{2(1-\varkappa)Q}\left(1 + \frac{a_1}{\omega} + \frac{a_2}{\omega^2}\right)$$

When $\zeta \approx a$, we have in accordance with (2) $\omega \approx M\Omega_a(a) \ln(a-\zeta)$. Taking this into consideration and restricting the last expansion to its principal term, we otain the formula

$$z \approx -\frac{\omega^2}{2(1-\varkappa)Q} \quad \text{for} \quad \zeta \approx a \tag{19}$$

Thus at a reasonable distance from the channel the effects of its geometrical details on the stream structure are smoothed out, and because of (19) the stream there is close to a zero-head one over the watertight stratum in the direction of a horizontal drain slit ([2], Sect. II). It was noted there that the hydrodynamic model of such flow coincides with the hydraulic one in some of its characteristics (the parabolic shape of the depression curve and the formula for the flow rate). The considered here flow fits with increasing accuracy in the hydraulic theory with increasing distance from the channel. Thus for the left-hand branch of the depression curve we obtain in the first approximation

$$y^2 \approx 2(1-\varkappa)Q|x| \quad \text{for} \quad \zeta \approx a \tag{20}$$

which is in agreement with (19).

The course of derivation of Eq. (17) shows that when $\zeta \to a$ the length of the horizontal projection of equipotential lines tends to the constant limit $(1-\varkappa)Q/2$. From this with allowance for (20) we conclude that with increasing distance from the channel the equipotential lines approach vertical straights.

We note certain particular cases.

1°. The symmetric flow. This case is derived from the considered above when $\varkappa = 1/2$, $a = a'$, $b = b'$, $m = m$ and $r = r'$ (the second equality is the corollary of the first and of (4)). It follows from (8) that $f = 0$. Formulas (10), (12) and (16) are identical to formulas (9), (11) and (15), respectively, while equality (13) is identically satisfied. Functions $\Omega(\tau)$ and $W(\tau)$ which appear in formulas (2) and (7) now assume the form

$$\Omega(\tau) = \frac{1}{(a^2-\tau^2)\sqrt{\tau^2-1}}, \quad W(\tau) = \frac{\tau(\tau^2-m^2)(\tau^2-r^2)}{(\tau^2-a^2)(\tau^2-1)^{1/2-\theta}(\tau^2-b^2)^{1+\theta}} \tag{21}$$

For the six remaining unknown parameters M, N, a, b, m and r we have the system of Eqs. (3), (9), (11), (14), (15) and (17). The last of these can be replaced by a simpler one which determines segment DE that in this case lies on the axis of symmetry which is the axis of ordinates. On the basis of (2), (7) and (21) and taking into account that on DE, $\zeta = i\sigma$, $0 \leqslant \sigma < \infty$, we obtain

$$NQ \int_0^\infty \frac{(m^2+u)(r^2+u)}{(a^2+u)(1+u^2)^{1/2-\theta}(b^2+u)^{1+\theta}} \times \left[\operatorname{arch} a - \operatorname{arch}\left(a\sqrt{\frac{1+u}{a^2+u}}\right)\right] du = 2\pi T$$

2°. The one-sided inflow. This scheme may be considered as the limit of the flow asymmetry in which the filtration flow rate of the channel depends exclusively of inflow from one side, for instance, from left, although part of the stream in the neighborhood of the watertight stratum passes then under the channel and reaches it only in the reverse motion. The stream dividing line DE coincides with the right-hand branch of the depression curve $A'B'C'$; as the limit streamline that branch becomes now an extension of the watertight stratum. Since points E and M' coincide with point A', region ω is converted into a half-band and the slit $A'M'E$ in region $1/w$ vanishes.

Formulas for functions ω, $1/w$ and z are derived from corresponding relationships established in the general case by passing to limit for $\varkappa \to 0$, and $a' \to \infty$. As the result we have

$$M = \frac{Q\sqrt{a^2-1}}{\pi}, \quad \Omega(\tau) = \frac{1}{(a-\tau)\sqrt{\tau^2-1}} \tag{22}$$

$$W(\tau) = \frac{(\tau-m)(\tau-r)(\tau+r')(\tau-f)}{(\tau-a)(\tau^2-1)^{1/2-\theta}[(\tau-b)(\tau+b')]^{1+\theta}}$$

In the system of equations defining parameters Eq. (10) loses its validity, the first of equalities (22) supersedes (3), and formula (8) becomes

$$\pi N[a - m - r + r' + f - (1+\theta)(b - b')] = 1$$

In remaining equations there are no coefficients containing parameters a' and m', and $\varkappa = 0$ is to be set in (17).

In accordance with (2) and (22) the following relationship

$$y(\zeta) = -\varphi(\zeta) + H + h_k = \frac{Q}{\pi}\operatorname{arch}\frac{a\zeta+1}{\zeta+a} + H + h_k, \quad b' \leqslant \zeta < \infty$$

is satisfied along the right-hand branch $A'B'$ of the depression curve, where $p = p_a - h_k$, hence the branch has in this case a horizontal asymptote $y = y_\infty$, and

$$y_\infty = \lim_{\zeta\to\infty} y(\zeta) = \frac{Q}{\pi}\operatorname{arch} a + H + h_k \tag{23}$$

It can be further shown that along $A'B'$ at some distance from the channel $y_\infty - y \approx \alpha_0 e^{-\alpha_1 x}$ (α_0 and α_1 are positive constants), i.e. the depression curve slope decreases and the flow is virtually absent.

It is expedient to substitute for Eq. (17) a simpler formula relating to the finite increment $y_\infty + T$ of the ordinate y at the circumvention of point A' to the corresponding increment of the integral in formula (7). Such relationship may be presented in the form

$$QN\sqrt{a^2-1} = \frac{Q}{\pi}\operatorname{arch} a + T + H + h_k$$

3°. Draining of a stratum of unbounded thickness. When the water tight stratum lies very deeply so that its effect on the flow to the drainage channel can be neglected, we have the scheme of a stratum of unbounded thickness which is obtained by passing to limit $a, a' \to \infty$. We assume a uniform flow of ground water to the drain from all directions, i.e. that the flow is symmetric about the y-axis and $\varkappa = 1/2$. We thus have a variant of the particular case considered in 1° for $a = \infty$. The solution is of the form

$$\omega = -\frac{Q}{\pi}\int_1^{\zeta}\frac{d\tau}{\sqrt{\tau^2-1}} + iQ = -\frac{Q}{\pi}\operatorname{arch}\zeta + iQ \tag{24}$$

$$z = N\int_1^{\zeta} W(\tau)\,[\omega(\zeta)-\omega(\tau)]\,d\tau, \quad W(\tau) = \frac{\tau(\tau^2-r^2)}{(\tau^2-1)^{1/2-\theta}(\tau^2-b^2)^{1+\theta}}$$

For the three parameters N, b and r we have the following system of equations (to which in the general case correspond Eqs. (11), (14) and (15)):

$$N\sin\pi\theta\left[\int_1^b \frac{W_b(\tau)-W_b(b)}{(b-\tau)^{1+\theta}}\,d\tau - \frac{W_b(b)}{\theta(b-1)^{\theta}}\right] = 1 \tag{25}$$

$$W_b(\tau) = \frac{\tau(\tau^2-r^2)}{(\tau^2-1)^{1/2-\theta}(b+\tau)^{1+\theta}}$$

$$NQ\int_0^1 \frac{\tau(r^2-\tau^2)\arcsin\tau}{(1-\tau^2)^{1/2-\theta}(b^2-\tau^2)^{1+\theta}}\,d\tau = \pi$$

$$N\sin\pi\theta\int_1^b \frac{W_b(\tau)(\operatorname{arch} b - \operatorname{arch}\tau)}{(b-\tau)^{1+\theta}}\,d\tau = \operatorname{arch} b + \frac{\pi}{Q}(H+h_k)$$

The third equation of system (25) may be used for the determination of parameter b after elimination from it of N and r with the use of the first two equations.

At some distance from the drain, i.e. for considerable ζ, we have in accordance with (24) the following relationships:

$$\omega(\zeta) \approx -(Q/\pi)\ln\zeta, \quad W(\tau) = 1 + O(\tau^{-2}).$$

$$\frac{dz}{d\omega} = \left(\frac{1}{w}\right)_{\zeta=\zeta_0} + N\int_{\zeta_0}^{\zeta} W(\tau)\,d\tau = N_0 + N\zeta + O\left(\frac{1}{\zeta}\right) \approx N_0 - Ne^{-\pi\omega/Q}$$

and, consequently,

$$z \approx N_0\omega + \frac{NQ}{\pi}e^{-\pi\omega/Q} \tag{26}$$

For the right-hand branch of the depression curve along which $\omega = \varphi = -y$ this formula becomes

$$x + iy \approx -N_0 y + \frac{NQ}{\pi}e^{\pi y/Q}$$

which implies that $N_0 = -i$; separation of the real part yields for that branch

$$y \approx \frac{Q}{\pi}\ln\frac{\pi x}{NQ} \approx \frac{Q}{\pi}\ln x \tag{27}$$

The comparison of (27) with (20) shows that at a fairly considerable distance from the channel the parabolic asymptotics of the depression curve branches is superseded by a logarithmic one when the depth of the watertight stratum increases. This reduces the rate of increase of the depression curve ordinates, which evinces the supporting action of the watertight stratum on the filtration flow. The effect of the watertight stratum can be appraised by using the scheme of one-sided inflow. Such scheme with essentially different flow properties in the neighborhood of points A and A' is only possible when the watertight stratum lies at a finite depth when these two points are separated. This separation vanishes with increasing depth of that stratum and, in accordance with (23), we have $\lim y_{\infty} = \infty$ when $a \to \infty$, which means that the ordinate of the right-hand branch of the depression curve increases indefinitely. At the limit $T \to \infty$ the pressure heads on the left and right of the drain tend to become equal at infinity (since they are both infinitely great), and the stream becomes two-sided and symmetric.

In concluding we would point out two hydrodynamic models similar to the one considered here, which are used in [3, 4]. In the first of these [3] the symmetric flow of ground water to a trapezoidal channel with permeable slopes with zero depth of water in it was considered. The second model described in [4] differs from the present one only by that instead of a channel it has a horizontal Joukowsky slit. Such difference should be important only in a certain neighborhood of the drain. If one takes into consideration that the solution obtained in [4] is considerably simpler than the solution derived here, it would appear that after appropriate correction it could be useful in calculations of the described flow.

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